The Mathematical Look at a Notion of the Compromise and its Ramifications

Aleksandar Hatzivelkos
University of Aplied Sciences Velika Gorica
Department of Mathematics
Zagrebačka 5, 10 410 Velika Gorica, Croatia
hatzi@vvg.hr

Abstract. *This paper aims to define a measure which would capture notion of compromise on a given profile of voter preferences, about certain candidate being appointed to the certain position by some social welfare function. The goal is to define what compromise should mean, and proposes so called "d-measure of divergence" as a measure of divergence for some candidate to be positioned to certain position. Two well established social welfare functions, Borda and plurality count d-measure of divergence results are compared. Furthermore, d-measure of divergence enables us to define new social welfare functions, which would (in different ways) minimize d-measure of divergence (from the compromise): "Simple d-measure", greedy minimization and "Total d-measure" social welfare functions. Characterization for the SdM function is given, and properties of later two functions are analyzed.*

Keywords. Borda count, Plurality count, Social choice function, Social welfare function, Compromise

1 Motivation and basic definitions

Main inspiration for this article comes from following example. Let there is an election in which one hundred voters should choose between three candidates: A, B and C. Each voter places its vote by ordering those candidates. That ordering we will call a *preference*, and denote it α_i. Set of all preferences for those hundred voters, a *profile* α is given in Table 1, in which fifty one voters have preference $A \succ B \succ C$, while forty nine voters have preference $C \succ B \succ A$.

Table 1: Basic motivation profile

51	49
A	C
B	B
C	A

Given the profile α, which candidate should win? Most of the classical social choice functions would say - candidate A. Borda count would produce $A \succ B \succ C$ linear ordering, result of plurality count would be $A \succ C \succ B$ (with A winning the most first places, and B the least). Condorcet method would duel all candidates, and those duels would yield $A \succ B \succ C$ ordering. All those classical methods have one thing in common: winner should be candidate A. Nevertheless, that is a candidate that 49% of voters see as the worst choice. Should A then be a winner? What should be result if we approach to a profile α looking for *compromise*? If we want from social choice function to address notion of compromise, wouldn't it be better if candidate B is declared as a winner?

This leads us to the main topic of this article: finding a way for determining a value which should capture notion of compromise on a given profile, for placing a candidate on a certain position in linear ordering. Let us go back to the example in Table 1. If we take a look at candidate A, in a given profile he/she is placed first in 51 preferences, and placed third in 49 preferences. Therefore, in 51 preferences, distance between his position and the first place is 0 (places), and in 49 preferences that distance equals 2 (places). If we simply sum all those distances (for each candidate) over profile α, we would get a measure of distance between profile placements of a candidate and a first place. But, for a such measure, one can easily prove that ranking based on it gives result equivalent to Borda count.

What we want to do is to put some weight on those distances, so that bigger distance carries more than just its linear contribution. Therefore, we will take look on a distance with some power d, d being a real number greater than 1. If we sum such weighted distances from the first place over profile α for a candidate A, we get following expression:

$$51 \cdot 0^d + 49 \cdot 2^d, \qquad d > 1$$

We introduce notion $\beta_1^d(A)$ for such expression. For other two candidates we have:

$$\beta_1^d(B) = 51 \cdot 1^d + 49 \cdot 1^d = 100$$

$$\beta_1^d(C) = 51 \cdot 2^d + 49 \cdot 0^d = 51 \cdot 2^d$$

Value $\beta_1^d(M_i)$ for some candidate M_i we will call *a d-measure of divergence from the first position*. The idea is that smaller value of $\beta_1^d(M_i)$ captures notion of the greater level of compromise on a given profile for a candidate to be placed on a first place of linear ordering. Unlike distance function from works of (Seiford, Cook, 1978), we do not form measure of distance between preferences. Rather than that, we establish measure divergence from compromise (or consensus) that certain candidate should be positioned on certain position. But the goal is similar: it is a in society interest to minimize that measure. This leads us to the following basic definitions.

1.1 Basic definitions

Definition 1.1 (d-Measure of divergence from the first position)**.** *Let $M = \{M_1, ..., M_m\}$ be set of m candidates, and let $\alpha \in \mathcal{L}(M)^n$ be a profile of n voters over those candidates. We define* a d-measure of divergence from the first position *for a candidate M_k, $\beta_1^d(M_k)$, as a*

$$\beta_1^d(M_k) = \sum_{i=1}^{n} \left|\alpha_i^k - 1\right|^d \tag{1}$$

where α_i^k stands for a position of the candidate M_k in a preference of i-th voter α_i, and for some real value $d > 1$.

Now we can easily extend definition to a d-measure of divergence from a j-th position of the k-th candidate:

Definition 1.2 (d-Measure of divergence from the j-th position)**.** *Let $M = \{M_1, ..., M_m\}$ be set of m candidates, and let $\alpha \in \mathcal{L}(M)^n$ be a profile of n voters over those candidates. We define* a d-measure of divergence from a j-th position *for a candidate M_k, $\beta_j^d(M_k)$, as a*

$$\beta_j^d(M_k) = \sum_{i=1}^{n} \left|\alpha_i^k - j\right|^d \tag{2}$$

where α_i^k stands for a position of the candidate M_k in a preference of i-th voter, α_i, and for some real value $d > 1$.

Given the Definition 1.2, it is only natural to gather $\beta_j^d(M_k)$ values in a form of a matrix:

Definition 1.3 (d-Measure of divergence matrix)**.** *Let $M = \{M_1, ..., M_m\}$ be set of m candidates, and let $\alpha \in \mathcal{L}(M)^n$ be a profile of n voters over those candidates. We define* a d-measure of divergence matrix

$$M^d = \begin{bmatrix} \beta_1^d(M_1) & \beta_2^d(M_1) & \cdots & \beta_m^d(M_1) \\ \beta_1^d(M_2) & \beta_2^d(M_2) & \cdots & \beta_m^d(M_2) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^d(M_m) & \beta_2^d(M_m) & \cdots & \beta_m^d(M_m) \end{bmatrix} \tag{3}$$

for some real value $d > 1$.

As we can see, in j-th column of a matrix M^d we have d-measures of divergence from j-th position for all candidates, while in i-th row of matrix M^d, we have d-measures of divergence from all positions for a candidate M_i.

1.2 Compromise as a Sorties paradox

Before we go further, let us say something about the value of parameter d. In the example given in the Table 1 we see that, if we want candidate B to have smaller d-measure of divergence from a first position than candidate A (which means, if we want candidate B to be declared a compromise winner on a given profile), it should be

$$\beta_1^d(B) < \beta_1^d(A) \qquad \Rightarrow \qquad d > \log_2 100 - \log_2 49$$

But, what is the value of d that should be used generally? Answering to that question requires finding an answer to the following version of a Sorties paradox (Hyde, 2014): Let us say that n voters are voting through strict linear ordering over the set of three candidates, $\{A, B, C\}$. For some $k \in \mathbb{N}$, $\lceil \frac{n}{2} \rceil \leq k \leq n$ they form profile α_{basic} given in Table 2.

Table 2: Basic definition profile

k	$n-k$
A	C
B	B
C	A

Let us accept reasoning that candidate B should be a compromise winner on profile α_{basic} given in Table 2 for $k = \lceil \frac{n}{2} \rceil$, and that candidate A should be a compromise winner on profile α_{basic} for $k = n$. Sorties paradox arises from question: what is the value of k for which we should consider candidate B a compromise winner on a given profile α_{basic}, while for value $k+1$ a compromise winner should be candidate A?

This question can be seen as a version of a classical Sorties paradox given by Megarian logician Eubulides of Miletus about a number of grains which are (not) forming a heap. Phenomenon that lies at a heart of the paradox is recognized as the phenomenon of vagueness; the concept of heap appears to lack sharp boundaries, just as the concept of compromise winner does in our case. Nevertheless, we will approach to the issue not as to a paradox, but (same as Eubulides did) as a puzzle.

From statement that for a value of $k = \lceil \frac{n}{2} \rceil$ we have one candidate as a compromise winner on a given profile, while for a value of $k = n$ we have another candidate as a compromise winner, using the Least-number principle, we shall conclude that there is some k_0 between those two values, such that for k_0 compromise winner on a profile α_{basic} is a candidate B, and for

$k_0 + 1$ compromise winner is candidate A. Value k_0 should be result of an apriori social choice of a group which is about to use this model.

There are numerous situations in which similar social choices are being made. For instance, in a number of states two-third parliament majority is needed to make constitution changes, as opposed to simple majority needed for other type of decisions. So, why 2/3? Why not 3/4 or 4/7? Measure of majority needed for such constitutional changes represents similar social decision, as one presented in this paper – society decided where to "draw the line". Similarly, social decision should be made about value of k_0. When value for k_0 is determined, for d we have:

$$\log_2\left(\frac{n}{n-k_0}\right) < d < \log_2\left(\frac{n}{n-k_0-1}\right)$$

2 d-Measure of divergence from the first position

If we interpret d-measure of divergence from the first position as a measure of compromise for a social function choice winner selection, we can compare results of the classical social choice function. For instance, Borda count is usually considered as a social choice function that emphasize compromise candidate as a winner, especially when compared to the plurality winner. Does this thesis holds if we use d-measure of divergence from the first position as a measure for selection of the compromise candidate for a winner?

In this section we will provide an answer to that question. To do that, we will first consider three candidate scenario, followed by scenarios with more candidates. Let us consider following theorem:

Theorem 2.1. *Let α be a profile over the set of candidates $M = \{A, B, C\}$. Let W_{BC} stands for a unique Borda count winner candidate and W_{PC} for a unique plurality winner candidate (if there are such) over some profile α. For every $d > 1$ there is*

$$\beta_1^d(W_{BC}) \leq \beta_1^d(W_{PC}).$$

Equality holds iff $W_{BC} = W_{PC}$.

There is combinatorial proof of this theorem. Although there are six different preferences over the set of three candidates, number of all possible combinations of preferences that can form a profile can be reduced.

Table 3: Condorcet triplets: profiles α_1^C and α_2^C

1	1	1
A	C	B
B	A	C
C	B	A

1	1	1
C	A	B
B	C	A
A	B	C

Two profiles α_1^C and α_2^C (Table 3) we will call Condorcet triples. Those profiles consist of three preferences on which neutral and anonymous social choice function should form a tie (or a cycle) as a result. As Saari showed (Saari, 1994), all scoring point functions are invariant in regard of Condorcet triplet removal, which includes both Borda and plurality count. Furthermore, it is easy to prove that a d-measure of divergence from the j-th position preserves ordering among candidates when profile is reduced for the Condorcet triple.

Lemma 2.2. *Let A and B be any two candidates from the set of candidates M. For a d-measure of divergence from the j-th position β_i^d, and profiles α and α', α' being the profile derived from α by removal of one Condorcet triple, we have*

$$\beta_i^d(A, \alpha) < \beta_i^d(B, \alpha) \quad \textit{iff} \quad \beta_i^d(A, \alpha') < \beta_i^d(B, \alpha').$$

Since both Borda and plurality count, as well as d-measure of divergence from the j-th position relations among candidates, are Condorcet triple invariant, we can reduce a set of all possible combinations of voters preferences to a set of preferences without Condorcet triples. This means that largest profile we should analyze consists of (some number of) two preferences from α_1^C and two from α_2^C. Still, it is a large work, and here we will demonstrate it on one of such reductions.

Lemma 2.3. *Let α_1 be a profile over the set of candidates $M = \{A, B, C\}$, without any Condorcet triples. Furthermore, let α_1 be a profile of a form*

k	l	m
$\xi(A)$	$\xi(C)$	$\xi(A)$
$\xi(B)$	$\xi(A)$	$\xi(C)$
$\xi(C)$	$\xi(B)$	$\xi(B)$

ξ being some permutation over the set M. Then, for every profile α obtained as a union of α_1 and some number of copies of Condorcet triples α_1^C and α_2^C it follows

$$\beta_1^d(W_{BC}) < \beta_1^d(W_{PC}),$$

for some $W_{BC}, W_{PC} \in M$, winners by Borda and plurality count respectively over the profile α, where $W_{BC} \neq W_{PC}$ if such W_{PC} exists.

Proof: Without loss of generality, we can drop permutation function ξ. On a given profile, plurality winner can be candidate A or C. If A is plurality winner, it follows $k + m > l$. For A to be Borda winner it should have greater Borda score than C, so $2k + l + 2m > 2l + m$, ie. $2k + m > l$ which holds beacuse of plurality winning condition. Therefore, if A is plurality winner, then it is also Borda winner, which proves the Lemma.

On the other hand, if C is plurylity winner, then we have:

$$k + m < l. \tag{4}$$

For A to be Borda winner (compared to C), we have

$$2k + l + 2m > 2l + m \;\Rightarrow\; 2k + m > l. \quad (5)$$

Conditions (4) and (5) sums to $m + k < l < m + 2k$. For a profile α_1 and $d > 1$ for which d-measure of divergence form the first place for a candidate C is smaller than the one for the candidate A, must hold:

$$\beta_1^d(C) < \beta_1^d(A) \quad \Leftrightarrow \quad m + k \cdot 2^d < l.$$

But from the condition (5) follows: $m + k \cdot 2^d > m + 2k > l$, which leads to the conclusion that C can not have smaller d-measure of divergence from the first place than candidate A, which proves the Lemma. □

Similar lemmas can be proven for all combination of preferences that can form profile α' which does not contain any Condorcet triple. Together with invariance of Borda count, plurality count and d-measure of divergence from the first position, in regard to removal (or addition) of Condorcet triplets, follows the proof of Theorem 2.1.

In case with more than three candidates, similar claim can not be proven. If there is four candidates, there are profiles on which plurality count produces different winner than Borda count, and with smaller d-measure of divergence form the first position. Consider the following theorem:

Theorem 2.4. *Let $M = \{A, B, C, D\}$ be set of candidates, and let W_{BC} stands for a unique Borda count winner candidate and W_{PC} for a unique plurality winner candidate over some profile α. For every $d > 1$ there is a profile α such that $W_{BC} \neq W_{PC}$, and for d-measures of divergence from the first position there is*

$$\beta_1^d(W_{PC}) < \beta_1^d(W_{BC}).$$

It can be proven that for a profile given in the Table 4 and a given $d > 1$, there are values for m, i and j for which Borda winner (candidate B) has a greater d-measure of divergence from the first place than the plurality winner (candidate A). In such profile, values for m, i and j are determined by:

$$i > \frac{2 \cdot 3^d - 2 \cdot 2^d - 1}{3^d - 2 \cdot 2^d + 1},$$

$$\frac{2^d - 2}{3^d - 2^d - 1} \cdot i + \frac{1}{3^d - 2^d - 1} < j < i - 1$$

$$\text{and} \qquad m = 2i - j - 1.$$

Let us demonstrate construction of the profile on which Borda winner has a greater d-measure of divergence from the first position than a plurality winner in the following example:

Example 2.5. *Let $M = \{A, B, C, D\}$ be set of candidates, and let* d $= 1.05$. *According to Theorem 2.4, values m, i and j from a profile in Table 4, equal to* $m = 44$, $i = 43$ *and* $j = 44$. *On this profile plurality winner is candidate A, while Borda winner is candidate B (with Borda score 217, while Borda scores of candidate A and C equals to 216 and 212). d-Measures of divergence from the first place equals to:*

$$\beta_1^{1.05}(A) = (43 + 41) \cdot 2^{1.05} = 173.9245$$

$$\beta_1^{1.05}(B) = 44 + 41 \cdot 3^{1.05} = 173.9455$$

Table 4: Construction profile for Theorem 2.4

m	i	j
A	B	D
B	C	C
C	A	A
D	D	B

Construction from the Theorem 2.4 can be expanded to the arbitrary large set of candidates. With first four positions of the candidates that remain the same, all other candidates can be arbitrary placed below 4th position in preferences of the constructed profile. Analysis of such profile leads to the same conditions for construction as in the case with four candidates.

3 d-Measure social welfare functions

3.1 Simple d-measure of divergence function, SdM

As defined in Section 1, a d-measure of divergence enables new approach to the construction of the social choice and social welfare functions. Simplest, and the most natural way to use information about d-measure of divergence, is to address a d-measure of divergence from first position. In most cases, it is only important who is the winner on a given profile. Therefore, we can define social welfare function based only upon d-measure of divergence from a first position, ie. values in the first column of a d-measure of divergence matrix, M^d.

Definition 3.1. *Let $M = \{M_1, ..., M_m\}$ be set of m candidates, $d > 1$ real number, and let $\alpha \in \mathcal{L}(M)^n$ be a profile of n voters over those candidates. Let $M_\alpha^d = \left[\beta_j^d(M_i)\right]$ be a d-measure of divergence matrix. We define social welfare function $SdM : \mathcal{L}(M)^n \to \mathcal{L}(M)$ in following way: let us make strict linear ordering of d-measures of divergence from a first position for all candidates*

$$\beta_1^d(M_{i_1}) < \beta_1^d(M_{i_2}) < ... < \beta_1^d(M_{i_m}). \quad (6)$$

We define strict linear ordering implied by (6) as a result of a social welfare function SdM:

$$SdM(\alpha) = M_{i_1} \succ M_{i_2} \succ M_{i_3} \succ ... \succ M_{i_m}, \quad (7)$$

if such strict ordering exists. Otherwise, we say social welfare function SdC is undefined on a profile α.

Of course, social welfare function SdM induces a social choice function (we will use the same name for the function), as a function which declares a candidate with minimal d-measure of divergence from the first position as a winner on a given profile. Let us now see which properties does satisfy social choice function SdM.

It is fairly simple to prove that SdM is **anonymous** and **neutral** for all $d > 1$ (which means it symmetrically treats voters and candidates), and that social choice function SdM is **positively responsive** (**monotonic**) for all $d > 1$ (we say for a social choice function that it is monotonic, if a voters' increase of a candidate ranking in his / her preference can not prevent that candidate from winning an election).

There are two more properties of a SdM social choice function we want to point out, *reinforcement* and *continuity*. First of those two, reinforcement, Young calls *consistency*, and he proves that this property characterizes position scoring social choice functions. Later of two, continuity, is technical property needed for proof of position scoring social choice functions characterization (Young, 1975).

Definition 3.2. *Social choice function F satisfies* reinforcement *if, whenever we split the set of preferences α into two subsets α' and α'', and some candidate would win for both subsets, then it will also win for the full profile α:*

$$F(\alpha') \cap F(\alpha'') \neq \emptyset \quad \Rightarrow \quad F(\alpha) = F(\alpha') \cap F(\alpha'')$$

Definition 3.3. *Social choice function F satisfies* continuity *if, for disjoint profiles α' and α'', where is $F(\alpha') = M_1$, $F(\alpha'') = M_2$, there is a number $n \in \mathbb{N}$, such that for a profile α made (as a union) of n copies of profile α' and profile α'' we have $F(\alpha) = M_1$.*

Proposition 3.4. *Social choice function SdM satisfies **reinforcement** and **continuity** for all $d > 1$.*

Proof: Since SdM can have just one winner, we have to show that for some candidate $M_w \in \{M_1, ..., M_m\}$, $SdM(\alpha') = SdM(\alpha'') = M_w$, it follows $SdM(\alpha) = M_w$. For all $d > 1$ it follows

$$\beta_1^d(M_w, \alpha') = \min_{\text{j=1,...,m}} \beta_1^d(M_j, \alpha'),$$

$$\beta_1^d(M_w, \alpha'') = \min_{\text{j=1,...,m}} \beta_1^d(M_j, \alpha''),$$

where $\beta_1^d(M_j, \alpha)$ stands for a d-measure of divergence from a first position for a candidate M_j on profile α. On the other hand, for a d-measure of divergence from a first position for a candidate M_j on profile α, we have

$$\beta_1^d(M_j, \alpha) = \beta_1^d(M_j, \alpha') + \beta_1^d(M_j, \alpha'').$$

Now, it follows

$$\beta_1^d(M_w, \alpha) = \beta_1^d(M_w, \alpha') + \beta_1^d(M_w, \alpha'') =$$

$$= \min_{\text{j=1,...,k}} \beta_1^d(M_j, \alpha') + \min_{\text{j=1,...,k}} \beta_1^d(M_j, \alpha'') \leq$$

$$\leq \min_{\text{j=1,...,k}} \left(\beta_1^d(M_j, \alpha') + \beta_1^d(M_j, \alpha'')\right) = \min_{\text{j=1,...,k}} \beta_1^d(M_j, \alpha)$$

Therefore, M_w is SdM winner on a profile α.

To prove continuity, let us suppose that $SdM(\alpha') = M_1$, $SdM(\alpha'') = M_2$ for disjoint profiles α' and α''. Suppose $m_1, m_2 \in \mathbb{R}$ equals to corresponding d-measures of divergence from a first position for those two candidates:

$$\begin{aligned} m_1 &= \beta_1^d(M_1, \alpha') = \min_{\text{j=1,...,k}} \beta_1^d(M_j, \alpha') \\ m_2 &= \beta_1^d(M_2, \alpha'') = \min_{\text{j=1,...,k}} \beta_1^d(M_j, \alpha''). \end{aligned} \tag{8}$$

For all $d > 1$ we have that a d-measures of divergence from a first position over the profile α made of n copies of profile α' and profile α'', and for every candidate M_i there is:

$$\beta_1^d(M_i, \alpha) = \beta_1^d(M_i, \alpha'') + n \cdot \beta_1^d(M_i, \alpha').$$

Specially, for candidates M_1 and M_2 we have

$$\beta_1^d(M_1, \alpha) = \beta_1^d(M_1, \alpha'') + n \cdot m_1,$$

$$\beta_1^d(M_2, \alpha) = m_2 + n \cdot \beta_1^d(M_2, \alpha').$$

Now, we have to show, that there is a $n \in \mathbb{N}$, such that

$$\beta_1^d(M_1, \alpha) < \beta_1^d(M_2, \alpha)$$

$$\beta_1^d(M_1, \alpha'') + n \cdot m_1 < m_2 + n \cdot \beta_1^d(M_2, \alpha')$$

$$\beta_1^d(M_1, \alpha'') - m_2 < n \cdot \left(\beta_1^d(M_2, \alpha') - m_1\right)$$

Because of (8), it follows $k_1 = \beta_1^d(M_1, \alpha'') - m_2 > 0$ and α' for $k_2 = \beta_1^d(M_2, \alpha') - m_1 > 0$. Now, it is clear, that there is $n \in \mathbb{N}$ such that $k_1 < n \cdot k_2$, which proves continuity. □

With these properties, we are able to prove main theorem about SdM social choice function.

Theorem 3.5. *For all $d > 1$, social choice function SdM is positional score function over set of m candidates, with scoring vector $s = (0, -1, -2^d, ..., -(m-1)^d)$.*

Proof: Young gives characterization (Young, 1975) of the social choice functions F as a (generalized) positional scoring rule iff it satisfies anonymity, neutrality, reinforcement and continuity. Furthermore, it is easy to see that positional score for some candidate M_k over scoring vector $s = \left(0, -1, -2^d, ..., -(m-1)^d\right)$ is maximized iff d-measure of divergence from a first position $\beta_1^d(M_k) = \sum_{i=1}^{n} \left|\alpha_i^k - 1\right|^d$ from Definition 1.1 is minimized. □

From Theorem 3.5 we can conclude that there is a positional score function which produces a winner that has the smallest d-measure of divergence from a first position. But modeling social choice functions according to a d-measure of divergence doesn't end there. In fact, we got relatively simple result because we used only one parameter, a d-measure of divergence from a *first position*. This means that we used only data from the first column from a d-measure of divergence matrix M^d. Now, we will expand our approach, trying to interpret all M^d data through social choice (welfare) functions.

3.2 Greedy approach to a d-measure of divergence

Main goal in this section will be construction of social wellfare function through utilization of a d-measure of divergence from *all positions*, not just winning one. Probably the simplest way to do that is using greedy technique; first place in linear order we will assign to a candidate with smallest d-measure of divergence from a first position, second place we will assign to candidate with smallest d-measure of divergence from a second position (from the set of remaining candidates, of course), and so on. Although algorithm sounds reasonable, it can produce strange results. Let us examine following example.

Example 3.6. *Suppose α is a profile over the set of five candidates $M = \{A, B, C, D, E\}$ given with following table:*

38	3	10
A	E	B
B	B	C
C	A	A
D	C	D
E	D	E

and let $d = 2$. Similar example can be constructed for lower values of d, but in this case we use $d = 2$ for clarity.

In this case, candidate A is clear Borda, Condorcet and plurality winner. But, that candidate doesn't have the lowest d-measure of divergence from a first position. For $d = 2$, we have $\beta_1^2(A) = 52$, and $\beta_1^2(B) = 41$, so by SdM and greedy approach to the d-measure of divergence, candidate B will be selected as a winner. But what about other candidates in a result linear ordering of a social welfare function? Let us take look at a d-measure of divergence matrix:

$$M^d(\alpha) = \begin{bmatrix} 52 & 51 & 152 & 355 & \mathbf{660} \\ \mathbf{41} & \cancel{10} & \cancel{81} & \cancel{254} & \cancel{529} \\ 189 & \mathbf{50} & \cancel{13} & \cancel{78} & \cancel{245} \\ 351 & 175 & \mathbf{60} & \cancel{27} & \cancel{48} \\ 768 & 435 & 204 & \mathbf{75} & \cancel{3} \end{bmatrix}$$

In i-th column, lowest d-measure of divergence from the i-th place is marked bold. Therefore, result of greedy approach on this profile is ordering

$$B \succ C \succ D \succ E \succ A.$$

Intuitively, this result is just - wrong. More formally, this example clearly demonstrates lack of Pareto efficiency of greedy approach to d-measure of divergence. In a given profile, candidate A is placed higher than candidate D in all preferences. Yet, greedy approach positioned candidate A on the last place of linear ordering, and so even after candidate D.

There is a way to deal with this problem. Greedy method could be applied cumulatively. After picking a candidate with the lowest value of d-measure of divergence in first column of a matrix M^d, when one chooses a candidate for a second position in result ordering, one can choose candidate (among remaining ones) which minimize a sum of d-measures of divergence from first and second position. And so on.

If we apply cumulative greedy approach to a profile described in Example 3.6, in first column of the matrix M^d we would choose candidate B (with the lowest d-measure of divergence in the first column). After that, we would choose candidate A with cumulative d-measure of divergence in the first and the second columns equal to 103, and so on. If we proceed with that algorithm, we would end up with the final ordering $B \succ A \succ C \succ D \succ E$.

Stated such way, cumulative greedy approach offers a way of exploring the subject of social welfare functions build upon a d-measure of divergence, but we won't pursue it in this paper. Rather than that, we will try to find comprehensive method for minimizing a d-measure of divergence.

3.3 Total minimization of a d-measure of divergence

Instead of trying to minimize a d-measure of divergence from positions one by one, as we tried to do with greedy approach, we will now look at minimization of all d-measures of divergence at once. This means that we will look for linear ordering, among all possible linear orderings of candidates, which minimize sum of d-measures of divergence from all positions. Such unique ordering (if it exists) we will take as a result of Total d-Measure social welfare function, TdM.

Definition 3.7. *Let $M = \{M_1, ..., M_m\}$ be set of m candidates, $d > 1$ real number, and let $\alpha \in \mathcal{L}(M)^n$ be a profile of n voters over those candidates. Let $M_\alpha^d = \left[\beta_j^d(M_i)\right]$ be a d-measure of divergence matrix. Let $\mathrm{Sym}(\mathbb{N}_m)$ be set of all permutations of set $\{1, 2, ..., m\}$, with elements $\xi_i, i \in \{1, 2, ..., m!\}$. We define social welfare function $TdM : \mathcal{L}(M)^n \to \mathcal{L}(M)$ ("Total d-Measure") in following way: let ξ_{TdM} be unique permutation of the set $\{1, 2, ..., m\}$ such that*

$$\sum_{i=1}^{m} \beta_i^d \left(M_{\xi_{TdM}(i)}\right) = \min_{j=1,\dots,m!} \sum_{i=1}^{m} \beta_i^d \left(M_{\xi_j(i)}\right) \quad (9)$$

We define:

$$TdM(\alpha) = M_{\xi_{TdM}(1)} \succ M_{\xi_{TdM}(2)} \succ ... \succ M_{\xi_{TdM}(m)}.$$

If there is no such unique permutation that satisfies equation (9), we say TdM is undefined on a profile α.

Remark 3.8. *In order to simplify notation, for a sum over a order of candidates (defined with a permutation ξ) on a given profile, and for some value $d > 1$,*

$\sum_{i=1}^{m} \beta_i^d \left(M_{\xi_{TdC}(i)}\right)$, *we will write*

$$\beta^d_{sum}\left(M_{\xi(1)} \succ M_{\xi(2)} \succ ... \succ M_{\xi(m)}\right) \quad or \quad \beta^d_{sum}(\xi).$$

For instance, if in Example 3.6, ordering of the candidates which has minimal sum of the d-measures of divergence would be $A \succ B \succ C \succ D \succ E$ with total sum 178. Notice that this ordering differs from result of SdM (and greedy) social welfare function, while it is same as result of Borda count. It is clear that TdM social welfare function doesn't have to produce winner with the lowest d-measure of divergence from the first position.

Before we go any further, let us establish basic property of TdM, that is, its asymptotic behavior. Unlike many other social welfare functions, it is not obvious that d-measure minimization over all orderings will produce an ordering which extremely dominate in a profile. In a way, following proposition is a version of the Young continuity condition, which was proven for SdM.

Proposition 3.9. *Let $M = \{M_1, M_2, ..., M_m\}$ be set of candidates, and let α be a profile, in which k_1 out of n voters have preference*

$$M_{\xi(1)} \succ M_{\xi(2)} \succ ... \succ M_{\xi(m)} \tag{10}$$

for some permutation over the set of candidates ξ, while other k_2 voters have some other preferences over the same set of candidates. If on a profile α number of k_1 voters increase (and consequently n), then there is some $k_0 \in \mathbb{N}$ such that for every $k_1 > k_0$ the result of social welfare function TdM is preference (10) for all $d > 1$.

Proof: Profile α can be written in general form:

k_1	$k_{2,1}$	$k_{2,2}$	$\cdots$	$k_{2,i}$
$M_{\xi(1)}$	$M_{\xi_1(1)}$	$M_{\xi_2(1)}$	$\cdots$	$M_{\xi_i(1)}$
$M_{\xi(2)}$	$M_{\xi_1(2)}$	$M_{\xi_2(2)}$	$\cdots$	$M_{\xi_i(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_{\xi(m)}$	$M_{\xi_1(m)}$	$M_{\xi_2(m)}$	$\cdots$	$M_{\xi_i(m)}$

for some number of preferences i, $i \leq k_2$, such that $k_{2,1} + k_{2,2} + \cdots + k_{2,i} = k_2$, $k_{2,1}, k_{2,2}, \cdots, k_{2,i} \in \mathbb{N}$. Let us now look how d-measure of divergence matrix M^d_α changes as k_1 increases. Places in matrix determined by ξ (that is, $\beta^d_1(M_{\xi(1)})$, $\beta^d_2(M_{\xi(2)})$, ..., $\beta^d_m(M_{\xi(m)})$) do not depend of k_1, so they remain the same. All other elements of M_d increases linearly as k_1 increases. Therefore, at some point $\beta^d_{sum}(\xi)$ will become minimal sum among all sums over permutations of the set of candidates M, which proves the proposition. □

Since TdM derives its result from properties of overall orderings, and not from properties of individual candidates, it is only natural to expect that properties based on a relationship between candidates would not always hold. Monotonicity is first of such properties. Suppose that α is profile over three candidates, A, B and C, which contains ordering $B \succ C \succ A$, such that A is TdM winner. If we swap positions of candidates A and C in that ordering to gain new ordering $B \succ A \succ C$, we will denote new profile with α'. Let us examine changes on a d-measure of a divergence matrix:

$$\Delta^d = M^d_\alpha - M^d_{\alpha'} = \begin{bmatrix} -2^d+1 & -1 & 1 \\ 0 & 0 & 0 \\ 2^d-1 & 1 & -1 \end{bmatrix} \tag{11}$$

From Δ^d we can read out changes in β^d_{sum} for various orderings (in notation, δ^d_{sum}) when we change profile from α to α'. Specialy, we have:

$$\begin{aligned} \delta^d_{sum}(A \succ C \succ B) &= 2 - 2^d \\ \delta^d_{sum}(B \succ A \succ C) &= -2 \end{aligned} \tag{12}$$

We can see that β^d_{sum} is reduced for both orderings, but reduction is greater for the ordering $B \succ A \succ C$ (for $d < 2$). So, to prove that social choice function TdM is not monotonic we should construct described profile α for which difference between β^d_{sum} of orderings $A \succ C \succ B$ and $B \succ A \succ C$ is lesser than difference between corresponding values of δ^d_{sum}.

In Table 5 non-monotonic change in one such profile (for d=1.15) is given; in the first profile TdM winner is candidate A, and after the change (in which candidate A swaps with candidate C for a higher position in one preference), TdM winner becomes candidate B.

Table 5: Non-monotonic profile for TdM and d=1.15

5	14	11
B	A	B
C	C	A
A	B	C

4	14	12
B	A	B
C	C	A
A	B	C

Example in Table 5 came from the analysis of three candidates scenario, which can be (without loss of generality) reduced to three cases, with candidate A winning on some profile α: 1) in preference $B \succ C \succ A$, candidate A swaps places with B; 2) in same preference candidate A swaps places with C; and 3) in preference $B \succ A \succ C$, candidate A swaps places with B. Analysis of the first case leads to conclusion that candidate A remains TdM winner.

Second case is described prior to Table 5. To draw a more general conclusion for all $d > 1$, we will analyze if there is a profile α such that (see (12)) difference $\delta^d_{sum}(A \succ C \succ B) - \delta^d_{sum}(B \succ A \succ C) = 4 - 2^d$ is greater than the difference $\beta^d_{sum}(B \succ A \succ C) - \beta^d_{sum}(A \succ C \succ B)$. We seek such a profile only for $1 < d < 2$, because in case of $d \geq 2$ candidate A remains winner.

Analysis of that scenario shows that such profile can be constructed for all $d \in \langle 1, 2 \rangle$, with construction rules similar to preference relations described in (Saari, 1994); each increase of certain preference in a profile does fixed change to the β^d_{sum}. This allows us to combine preferences in order to achieve certain distance in observed sums. We won't go into detail of that extensive analysis in this paper. Let us just point out that, values for d which are close to 2 yield non-monotonic profiles with much higher total number of preferences (voters). This can be interesting finding if we are considering usage of TdM social choice function in a smaller group of voters. Third case leads to similar conclusion, but this time for values of d greater than 2. Finding described profiles is relatively simple when value of d is close to 1 or very large, and it becomes more challenging when $d \to 2$. Even more, we have following result, which follows from analysis of a matrix Δ^d:

Proposition 3.10. *For d=2, social choice function TdM over set of three candidates is monotonic.*

Another important property of the social welfare functions is Pareto efficiency.

Theorem 3.11. *Social welfare function TdM is Pareto efficient for every value* $d > 1$.

Sketch of the proof: Suppose, TdM is not Pareto efficient, i.e. there is a profile α and two candidates A and B, such that is $A \succ B$ in all preferences of the profile, but $B \succ A$ in a result ordering of the TdM. Let us assume (without loss of generality) that candidate B is on the i-th position in the result ordering, and candidate A i the j-th position, $i < j$. Let us now analyze each of the preferences that form profile α. Since in each of those preferences A comes before B, we can assume that A is on the k-th position, and B is on the l-th position, with $k < l$. There are six different combinations of values i, j. k and l. For all of those combinations we analyze contribution of the analyzed preference to the β^d_{sum} for two linear orderings: one which by assumption minimizes β^d_{sum} over all preferences (with $B \succ A$ in the ordering), and one constructed from that ordering with swap of the positions for A and B. Analyzis shows that contribution to β^d_{sum} is smaller for the ordering which contains $A \succ B$, in all six preference types. □

4 Conclusion

In this paper we provided new approach to the notion of compromise on a given profile of voters preferences. For a compromise seen as a version of Sorties paradox, we provided definition of d-measure of divergence from the j-th position, which gives numerical value of divergence from a compromise that k-th candidate should be placed on j-th position in strict linear ordering of the candidates.

Even if Borda count is usually considered a social choice function which provides compromise result (especially when compared to plurality count), this statement holds only in case of three candidates scenario. If there are four or more candidates, there are profiles on which plurality count produces winner with lower d-measure of divergence from the first place than Borda count.

Next, d-measure of divergence enables us to define new social welfare functions, which would (in different ways) minimize d-measure of divergence (from the compromise): SdM, greedy minimization, and TdM social welfare functions. Characterization for the SdM function, as a scoring point social choice function is given. Since d-measure of divergence offers much richer ways for utilization, two more concept of d-measure of divergence minimization are introduced. But while greedy approach didn't produce encouraging results, there is a cumulative greedy approach which can lead to interesting results in future analyzes. Finally, we introduced concept of social choice function build upon total minimization of d-measure of divergence, and proved its basic properties: asymptotic behavior, lack of monotonicity and Pareto efficiency.

Acknowledgments

The research was supported by the University of Applied Sciences Velika Gorica.

References

Cook W. D., Seiford L. M.: *Priority ranking and consensus formation*, Manage Science 24(16), (1978), 1721-1732.

Hyde D., "*Sorites Paradox*", The Stanford Encyclopedia of Philosophy (Winter 2014 Edition), Edward N. Zalta (ed.).

Myerson, R. B. "*Fundamentals of Social Choice Theory*", Quarterly Journal of Political Science (2013): Vol. 8: No. 3, pp 305-337.

Saari, D. G., "*Geometry of Voting*", Springler-Verlag, New York (1994)

Young, H. P. *"Social Choice Scoring Functions"*, SIAM Journal on Applied Mathematics, Vol. 28, No. 4 , pp 824-838 (1975)